Exercises for Section 4.2

1. Apply Theorem 4.2.4 to determine the following limits:
   (a) $\lim_{x \to 1} (x+1)(2x+3) \quad (x \in \mathbb{R})$,
   (b) $\lim_{x \to 1} \dfrac{x^2+2}{x^2-2} \quad (x > 0)$,
   (c) $\lim_{x \to 2} \left(\dfrac{1}{x+1} - \dfrac{1}{2x} \right) \quad (x > 0)$,
   (d) $\lim_{x \to 0} \dfrac{x+1}{x^2+2} \quad (x \in \mathbb{R})$.
2. Determine the following limits and state which theorems are used in each case. (You may wish to use Exercise 15 below.)
   (a) $\lim_{x \to 2} \sqrt{\dfrac{2x+1}{x+3}} \quad (x > 0)$,
   (b) $\lim_{x \to 2} \dfrac{x^2-4}{x-2} \quad (x > 0)$,
   (c) $\lim_{x \to 0} \dfrac{(x+1)^2-1}{x} \quad (x > 0)$,
   (d) $\lim_{x \to 1} \dfrac{\sqrt{x}-1}{x-1} \quad (x > 0)$.
3. Find $\lim_{x \to 0} \dfrac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$ where $x > 0$.
4. Prove that $\lim_{x \to 0} \cos(1/x)$ does not exist but that $\lim_{x \to 0} x\cos(1/x) = 0$.
5. Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, and let c be a cluster point of A. Suppose that f is bounded on a neighborhood of c and that $\lim_{x \to c} g = 0$. Prove that $\lim_{x \to c} fg = 0$.
6. Use the definition of the limit to prove the first assertion in Theorem 4.2.4(a).
7. Use the sequential formulation of the limit to prove Theorem 4.2.4(b).
8. Let $n \in \mathbb{N}$ be such that $n \geq 3$. Derive the inequality $-x^2 \leq x^n \leq x^2$ for $-1 < x < 1$. Then use the fact that $\lim_{x \to 0} x^2 = 0$ to show that $\lim_{x \to 0} x^n = 0$.
9. Let f, g be defined on A to $\mathbb{R}$ and let c be a cluster point of A.
   (a) Show that if both $\lim_{x \to c} f$ and $\lim_{x \to c} (f+g)$ exist, then $\lim_{x \to c} g$ exists.
   (b) If $\lim_{x \to c} f$ and $\lim_{x \to c} fg$ exist, does it follow that $\lim_{x \to c} g$ exists?
10. Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f+g$ and fg have limits at c.
11. Determine whether the following limits exist in $\mathbb{R}$.
    (a) $\lim_{x \to 0} \sin(1/x^2) \quad (x \neq 0)$,
    (b) $\lim_{x \to 0} x\sin(1/x^2) \quad (x \neq 0)$,
    (c) $\lim_{x \to 0} \operatorname{sgn} \sin(1/x) \quad (x \neq 0)$,
    (d) $\lim_{x \to 0} \sqrt{x}\sin(1/x^2) \quad (x > 0)$.
12. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+y) = f(x) + f(y)$ for all x, y in $\mathbb{R}$. Assume that $\lim_{x \to 0} f = L$ exists. Prove that $L = 0$, and then prove that f has a limit at every point $c \in \mathbb{R}$. [*Hint*: First note that $f(2x) = f(x) + f(x) = 2f(x)$ for $x \in \mathbb{R}$. Also note that $f(x) = f(x-c) + f(c)$ for x, c in $\mathbb{R}$.]
13. Functions f and g are defined on R by $f(x) := x + 1$ and $g(x) := 2$ if $x \neq 1$ and $g(1) := 0$.
    (a) Find $\lim_{x \to 1} g(f(x))$ and compare with the value of $g(\lim_{x \to 1} f(x))$.
    (b) Find $\lim_{x \to 1} f(g(x))$ and compare with the value of $f(\lim_{x \to 1} g(x))$.
14. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If $\lim_{x \to c} f$ exists, and if $|f|$ denotes the function defined for $x \in A$ by $|f|(x) := |f(x)|$, prove that $\lim_{x \to c} |f| = \left| \lim_{x \to c} f \right|$.
15. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. In addition, suppose that $f(x) \geq 0$ for all $x \in A$, and let $\sqrt{f}$ be the function defined for $x \in A$ by $(\sqrt{f})(x) := \sqrt{f(x)}$. If $\lim_{x \to c} f$ exists, prove that $\lim_{x \to c} \sqrt{f} = \sqrt{\lim_{x \to c} f}$.

Section 4.3 Some Extensions of the Limit Concept†

In this section, we shall present three types of extensions of the notion of a limit of a function that often occur. Since all the ideas here are closely parallel to ones we have already encountered, this section can be read easily.

† This section can be largely omitted on a first reading of this chapter.

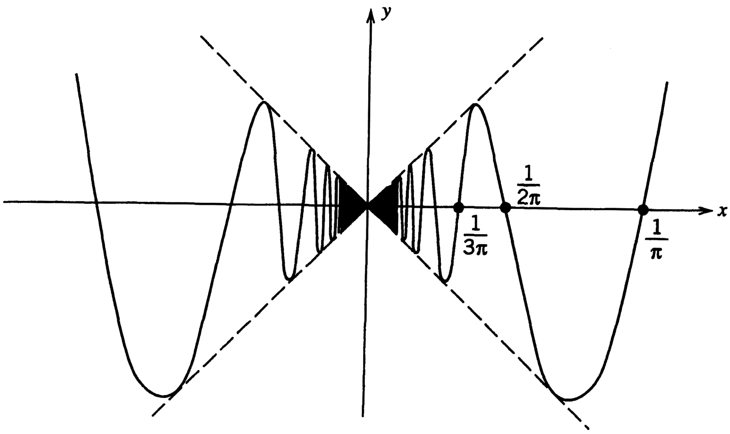

Figure 5.1.3 Graph of $f(x) = x \sin(1/x) \quad (x \neq 0)$

Exercises for Section 5.1

1. Prove the Sequential Criterion 5.1.3.
2. Establish the Discontinuity Criterion 5.1.4.
3. Let $a < b < c$. Suppose that f is continuous on $[a, b]$, that g is continuous on $[b, c]$, and that $f(b) = g(b)$. Define h on $[a, c]$ by $h(x) := f(x)$ for $x \in [a, b]$ and $h(x) := g(x)$ for $x \in [b, c]$. Prove that h is continuous on $[a, c]$.
4. If $x \in \mathbb{R}$, we define $[\![x]\!]$ to be the greatest integer $n \in \mathbb{Z}$ such that $n \leq x$. (Thus, for example, $[\![8.3]\!] = 8, [\![\pi]\!] = 3, [\![-\pi]\!] = -4$.) The function $x \mapsto [\![x]\!]$ is called the **greatest integer function.** Determine the points of continuity of the following functions:
   (a) $f(x) := [\![x]\!]$, (b) $g(x) := x[\![x]\!]$,
   (c) $h(x) := [\![\sin x]\!]$, (d) $k(x) := [\![1/x]\!] \quad (x \neq 0)$.
5. Let f be defined for all $x \in \mathbb{R}, x \neq 2$, by $f(x) = (x^2 + x - 6)/(x - 2)$. Can f be defined at $x = 2$ in such a way that f is continuous at this point?
6. Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ be continuous at a point $c \in A$. Show that for any $\varepsilon > 0$, there exists a neighborhood $V_\delta(c)$ of c such that if $x, y \in A \cap V_\delta(c)$, then $|f(x) - f(y)| < \varepsilon$.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$. Show that there exists a neighborhood $V_\delta(c)$ of c such that if $x \in V_\delta(c)$, then $f(x) > 0$.
8. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $S := \{x \in \mathbb{R} : f(x) = 0\}$ be the "zero set" of f. If (x_n) is in S and $x = \lim(x_n)$, show that $x \in S$.
9. Let $A \subseteq B \subseteq \mathbb{R}$, let $f : B \to \mathbb{R}$ and let g be the restriction of f to A (that is, $g(x) = f(x)$ for $x \in A$).
   (a) If f is continuous at $c \in A$, show that g is continuous at c.
   (b) Show by example that if g is continuous at c, it need not follow that f is continuous at c.
10. Show that the absolute value function $f(x) := |x|$ is continuous at every point $c \in \mathbb{R}$.
11. Let $K > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \mathbb{R}$. Show that f is continuous at every point $c \in \mathbb{R}$.
12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$ for every rational number r. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.
13. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) := 2x$ for x rational, and $g(x) := x + 3$ for x irrational. Find all points at which g is continuous.

14. Let $A := (0, \infty)$ and let $k : A \to \mathbb{R}$ be defined as follows. For $x \in A$, x irrational, we define $k(x) = 0$; for $x \in A$ rational and of the form $x = m/n$ with natural numbers m, n having no common factors except 1, we define $k(x) := n$. Prove that k is unbounded on every open interval in A. Conclude that k is not continuous at any point of A. (See Example 5.1.6(h).)
15. Let $f : (0, 1) \to \mathbb{R}$ be bounded but such that $\lim_{x \to 0} f$ does not exist. Show that there are two sequences (x_n) and (y_n) in (0, 1) with $\lim(x_n) = 0 = \lim(y_n)$, but such that $(f(x_n))$ and $(f(y_n))$ exist but are not equal.

Section 5.2 Combinations of Continuous Functions

Let $A \subseteq \mathbb{R}$ and let f and g be functions that are defined on A to $\mathbb{R}$ and let $b \in \mathbb{R}$. In Definition 4.2.3 we defined the sum, difference, product, and multiple functions denoted by $f + g, f - g, fg, bf$. In addition, if $h : A \to \mathbb{R}$ is such that $h(x) \neq 0$ for all $x \in A$, then we defined the quotient function denoted by f/h.

The next result is similar to Theorem 4.2.4, from which it follows.

5.2.1 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $b \in \mathbb{R}$. Suppose that $c \in A$ and that f and g are continuous at c.*

(a) *Then $f + g, f - g, fg$, and bf are continuous at c.*

(b) *If $h : A \to \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0$ for all $x \in A$, then the quotient f/h is continuous at c.*

Proof. If $c \in A$ is not a cluster point of A, then the conclusion is automatic. Hence we assume that c is a cluster point of A.

(a) Since f and g are continuous at c, then

$$f(c) = \lim_{x \to c} f \quad \text{and} \quad g(c) = \lim_{x \to c} g.$$

Hence it follows from Theorem 4.2.4(a) that

$$(f + g)(c) = f(c) + g(c) = \lim_{x \to c} (f + g).$$

Therefore $f + g$ is continuous at c. The remaining assertions in part (a) are proved in a similar fashion.

(b) Since $c \in A$, then $h(c) \neq 0$. But since $h(c) = \lim_{x \to c} h$, it follows from Theorem 4.2.4(b) that

$$\frac{f}{h}(c) = \frac{f(c)}{h(c)} = \frac{\lim_{x \to c} f}{\lim_{x \to c} h} = \lim_{x \to c} \left(\frac{f}{h} \right).$$

Therefore f/h is continuous at c. Q.E.D.

The next result is an immediate consequence of Theorem 5.2.1, applied to every point of A. However, since it is an extremely important result, we shall state it formally.

5.2.2 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be continuous on A to $\mathbb{R}$, and let $b \in \mathbb{R}$.*

(a) *The functions $f + g, f - g, fg$, and bf are continuous on A.*

(b) *If $h : A \to \mathbb{R}$ is continuous on A and $h(x) \neq 0$ for $x \in A$, then the quotient f/h is continuous on A.*